

# Is trivial the antiferromagnetic $\mathbf{RP}^2$ model in four dimensions?

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## Abstract

We study the antiferromagnetic  $\mathbf{RP}^2$  model in four dimensions. We find a second order transition with two order parameters, one ferromagnetic and the other antiferromagnetic. The antiferromagnetic sector has mean-field critical exponents and a renormalized coupling which goes to zero in the continuum limit. The exponents of the ferromagnetic channel are not the mean-field ones, but the difference can be interpreted as logarithmic corrections. We perform a detailed analysis of these corrections and conclude the triviality of the continuum limit of this model.

# 1 Introduction

The non-perturbative formulation of non-asymptotically free, interacting field theories in four dimensions is yet to be accomplished. The conventional analysis, for  $\lambda\phi^4$  and  $O(N)$  theories, yields triviality in four dimensions [1]. That is, once the continuum limit is taken, correlation functions factorize as the Wick's theorem prescribes for the gaussian theory. A possible way, in order to obtain a model with a non-trivial continuum limit, is to introduce antiferromagnetism (AFM). Gallavotti and Rivasseau have considered AFM actions to change the ultraviolet limit of  $\phi^4$  theories [2]. From a Statistical Physics point of view, a great variety of AFM models in three dimensions has been studied to obtain different qualitative behaviour from that of the corresponding ferromagnetic (FM) models [3]. In four dimensions, recent works have studied the possibility of new universality classes if AFM is added [4].

The AFM  $\mathbf{RP}^2$  model has recently been studied in three dimensions, due to its exotic properties [5, 6]. For instance, it has a disordered, unfrustrated ground state. Even more, it seems to present a full breaking of the action's  $O(3)$  symmetry [6]. Perturbative studies of this Spontaneous Symmetry Breaking (SSB) pattern, yield the  $O(4)$  universality class [8]. If this prediction holds true in four dimensions, the fate of the model is triviality.

However, this theoretical prediction has been questioned in three dimensions by Monte Carlo (MC) simulations [6]. Therefore, the study of the triviality of this model in four dimensions is very interesting. It also would help to enlighten the situation in three dimensions. We will see, although, that a detailed analysis of the MC simulation of this model indicates the triviality of its continuum limit through the appearance of logarithmic corrections to the divergences of the observables of the theory. A special form of the Finite Size Scaling (FSS) analysis which includes logarithmic corrections will be used to deal with these corrections.

We define the model and observables in section §2, where we also describe the techniques we have used to measure the critical exponents. The results of the MC simulation are presented in section §3. The model exhibits a phase transition at a negative coupling with two independent order parameters. One of this channels, the staggered one, presents mean-field critical exponents, but the other, ferromagnetic, presents deviations. We show in sections §3 and §4 how the discrepancies with the mean-field behaviour can be interpreted as logarithmic corrections.

## 2 The model

We shall consider the  $\mathbf{RP}^2 \equiv \mathbf{S}^2/\mathbf{Z}_2$  (real projective space) spin model in four dimensions. Our basic variable is a three component normalized spin  $\mathbf{v}_i$ , interacting through a gauge  $\mathbf{Z}_2$ -invariant action. As a local symmetry cannot be broken, these are effectively  $\mathbf{RP}^2$  variables (only the direction of the vectors is relevant). We consider a hypercubic lattice, with first neighbour coupling:

$$S = \beta \sum_{\langle ij \rangle} (\mathbf{v}_i \cdot \mathbf{v}_j)^2 \quad , \quad \mathcal{Z} = \int \left( \prod_i d\mathbf{v}_i \right) e^S \quad (1)$$

The ferromagnetic (positive coupling) model presents a first order transition at  $\beta \approx 0.94$ . The ground state consists of spins parallel or antiparallel to an arbitrary direction, and the SSB is  $\text{SO}(3)/\text{SO}(2)$ . The analysis of the antiferromagnetic counterpart is trickier, given the more complicated nature of the ground state. Let us call a lattice site, labeled by  $(x, y, z, t)$ , even or odd according to the parity of  $x + y + z + t$ . In the ground state, every even/odd spin is parallel or antiparallel to an arbitrary direction, while odd/even spins lie randomly on the perpendicular plane. The corresponding SSB is  $\text{SO}(3)/\text{SO}(2)$ , which calls for the  $\text{O}(3)$  universality class. However, fluctuations induce an interaction between spins on the randomly-plane sublattice. In three dimensions, this seems enough to break the remaining  $\text{O}(2)$  symmetry [6], and to change the Universality Class.

### 2.1 Definition of observables

The natural  $\mathbf{RP}^2$  variables are given by the traceless tensorial field  $\mathbf{T}_i$

$$\mathbf{T}_i^{\alpha\beta} = v_i^\alpha v_i^\beta - \frac{1}{3} \delta^{\alpha\beta}, \quad (2)$$

whose lattice Fourier transform will be represented by  $\hat{\mathbf{T}}$ :

$$\hat{\mathbf{T}}_{\mathbf{p}} = \sum_{\mathbf{r}} \exp(-i\mathbf{p} \cdot \mathbf{r}) \mathbf{T}_{\mathbf{r}}. \quad (3)$$

We will work in a  $L^4$  lattice with periodic boundary conditions. We define two order parameters, according to the discussion of the ground state above,

the intensive staggered (ferromagnetic) magnetization, as the sums of tensors on even sites minus (plus) those on odd sites, or equivalently

$$\mathbf{M}_s = \frac{1}{L^4} \hat{\mathbf{T}}_{(\pi, \pi, \pi, \pi)} \quad , \quad (\mathbf{M} = \frac{1}{L^4} \hat{\mathbf{T}}_{(0,0,0,0)}). \quad (4)$$

As no spontaneous symmetry breaking can occur on a finite lattice, in a MC simulation one needs to measure O(3)-invariant operators. For the magnetization and the susceptibility, we define

$$M = \left\langle \sqrt{\text{tr} \mathbf{M}^2} \right\rangle \quad , \quad \chi = L^4 \left\langle \text{tr} \mathbf{M}^2 \right\rangle, \quad (5)$$

and analogously with the staggered observables.

A very useful quantity for a triviality study is the Binder cumulant. For this model, we define

$$V_M = \frac{5}{2} \left( \frac{7}{5} - \frac{\langle (\text{tr} \mathbf{M}^2)^2 \rangle}{\langle \text{tr} \mathbf{M}^2 \rangle^2} \right), \quad (6)$$

which, in the infinite volume limit, becomes 1 in the broken phase and 0 in the symmetric one. The cumulant for the staggered magnetization is defined analogously.

Another very interesting quantity is the second momentum correlation length defined as [9]

$$\xi_L = \left( \frac{\chi/F - 1}{4 \sin^2(\pi/L)} \right)^{1/2}, \quad (7)$$

where  $F$  is the mean value of the trace of  $\hat{\mathbf{T}}$  squared at minimal momentum ( $2\pi/L$  in any of the four directions). For  $\xi_s$  we use  $\chi_s$  and  $F_s$ , analogously defined from  $\hat{\mathbf{T}}$  at momentum  $(2\pi/L + \pi, \pi, \pi, \pi)$  and permutations.

The field theoretical definition of the renormalized coupling constant can now be introduced

$$g_R = V_M (L/\xi_L)^d, \quad (8)$$

where  $d$  is the dimension of the lattice. We will consider the renormalized couplings associated with the two different sectors.

In addition, we measure the energy, which is needed for the spectral density method [10], invaluable for extrapolating MC measures to a neighbourhood of the critical coupling.

## 2.2 Standard Finite Size Scaling

To study critical exponents, we have used a method specially suited to the measurements of anomalous dimensions [6], [7]. Let us consider the mean value of an operator  $O$ , measured in a size  $L$  lattice, at a coupling value  $\beta$  in the critical region. Let  $t$  be the reduced temperature  $(\beta - \beta_c)/\beta_c$ . The standard FSS formula states that [11]

$$\langle O(L, t) \rangle = \langle O(t) \rangle \mathcal{F}_O(s(L, t)) , \quad s(L, t) \equiv \frac{L}{\xi(t)} , \quad (9)$$

where  $O(t)$  means  $O(\infty, t)$  and  $\mathcal{F}_O$  is a smooth function. We suppose that the values of  $L$  and  $\xi(t)$  are large, so that we ignore scaling corrections in (9). Now, we have  $\langle O(t) \rangle \sim t^{-x_O}$ , which is the definition of the critical exponent  $x_O$ , and  $\xi(t) \sim t^{-\nu}$ , and we can write  $L^{x_O/\nu} = \langle O(t) \rangle s^{x_O/\nu}$ . This allows to write (9) as

$$\langle O(L, t) \rangle = L^{x_O/\nu} \mathcal{G}_O(s). \quad (10)$$

Applying (10) to the correlation length, it gives  $\xi(L, t) = L \mathcal{G}_\xi(s)$ , so that

$$x(L, t) \equiv \frac{\xi(L, t)}{L} = \mathcal{G}_\xi(s), \quad (11)$$

and  $s = \mathcal{G}_\xi^{-1}(x)$ . From eq. (10) we have

$$\langle O(L, t) \rangle = L^{x_O/\nu} \mathcal{G}_O(\mathcal{G}_\xi^{-1}(x)) \equiv L^{x_O/\nu} f_O(x), \quad (12)$$

and we finish up with the useful expression

$$\langle O(L, t) \rangle = L^{x_O/\nu} f_O\left(\frac{\xi(L, t)}{L}\right) + \dots, \quad (13)$$

where the dots stand for possible scaling-corrections. Let us denote

$$Q_O = \frac{\langle O(rL, t) \rangle}{\langle O(L, t) \rangle}, \quad (14)$$

we can produce with it a sensible measure of critical exponents:

$$Q_O|_{Q_\xi=r} = r^{x_O/\nu} + \dots. \quad (15)$$

Therefore, from simulations of lattice sizes  $L$  and  $rL$ , we can extract the critical exponent  $x_O/\nu$  from the quotient (14), measured at the point where one correlation length is  $r$  times the other.

$L$	$\tau(\chi_s)$	Measures ( $\times 10^3 \tau(\chi_s)$ )
4	0.644(16)	30
6	1.189(20)	60
8	2.26(3)	64
10	3.65(11)	19
12	5.53(17)	21
16	10.7(2)	39
20	17.0(10)/2.31(11)	5/5
24	25.5(10)/3.53(17)	4/2

Table 1: Total number of measures and the corresponding integrated correlation times  $\tau$  for  $\chi_s$ . In the larger lattices the data of the overrelaxed simulations (right) are separated from those of Metropolis (left) by slashes.

### 3 The Monte Carlo simulation

To simulate the system, we have used a standard three-hits Metropolis algorithm, with an uncorrelated change proposal, achieving approximately a 50% of acceptance. The lattice sizes have been  $L = 4, 6, 8, 10, 12, 16, 20$  and  $24$ . For the larger sizes,  $20$  and  $24$ , we have combined Metropolis with an overrelaxed update, described in Appendix §A, to decrease the autocorrelation time. The overrelaxed algorithm is not able to decrease the dynamic critical exponent  $z$ , but nevertheless we save total CPU time when compared with the simple Metropolis simulation.

The runs have been distributed over several Workstations. We display in Table 1 the integrated autocorrelation time for  $\chi_s$  and the number of measurements performed for every lattice size. Every two measurements are separated by 10 sweeps, each consisting of either one Metropolis update or one Metropolis plus three overrelaxed updates when we use the latter algorithm.

#### 3.1 About the order parameters

The  $\mathbf{RP}^2$  model presents a second order phase transition at  $\beta \sim -1.34$ . The ferromagnetic and staggered magnetizations defined in equation (5) are zero

below the transition. To show that they are real order parameters, we should ensure that they do not vanish in the broken phase when  $V \rightarrow \infty$ . In Fig. 1 we plot the values of  $M_s$  and  $M$  at  $\beta = -1.5$  for the lattice sizes  $L = 8, 12$  and  $16$ . It is clear that both magnetizations reach an asymptotic value different from zero in the thermodynamical limit in the broken phase.

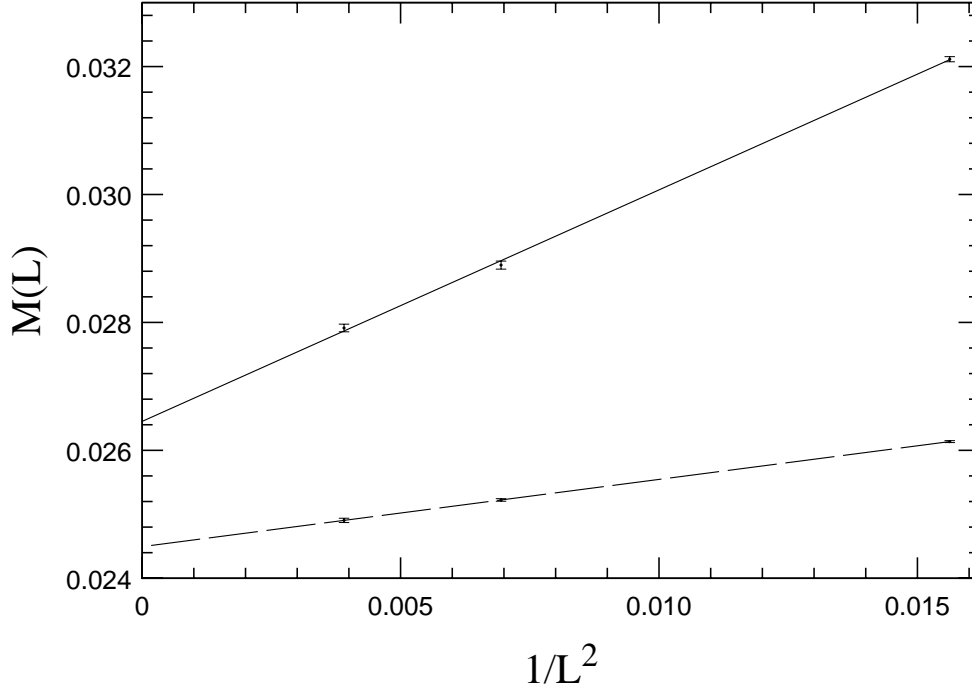


Figure 1: Asymptotic values of  $M$  (straight line) and  $M_s/10$  (dashed line) from their values for  $L = 8, 12$  and  $16$  at  $\beta = -1.5$ .

### 3.2 Critical exponents

We have calculated the critical exponents  $\nu$  and  $\eta$  for the two different channels using (15), which yields  $x_\chi = \gamma$  for the susceptibility, and  $x_M = -\beta$  for the magnetization. To calculate  $\nu$ , we use  $x_{\partial\xi/\partial\beta} = \nu + 1$ . All along this paper we shall take  $r = 2$ .

We obtain the anomalous dimension  $\eta$  through the scaling relations:

$$(2 - \eta)\nu = \gamma \quad , \quad 2\beta = \nu(d - 2 + \eta). \quad (16)$$

The resulting values for the  $\eta$  exponent from these two relations, will be denoted by  $\eta_\chi$  and  $\eta_M$  respectively.

As far as the exponent  $\nu$  is concerned, we expect the same critical exponent for both correlation lengths, the ferromagnetic  $\xi^{\text{FM}}$  and the staggered  $\xi$ . We have found that the measures for  $\xi$  are more accurate so we have used this variable as correlation length.

We plot in Fig. 2 an example of how this method works in both channels. Notice that  $Q_{M_{(s)}^2}$  takes the value  $2^{\gamma_{(s)}/\nu-d}$  when  $Q_\xi = 2$ .

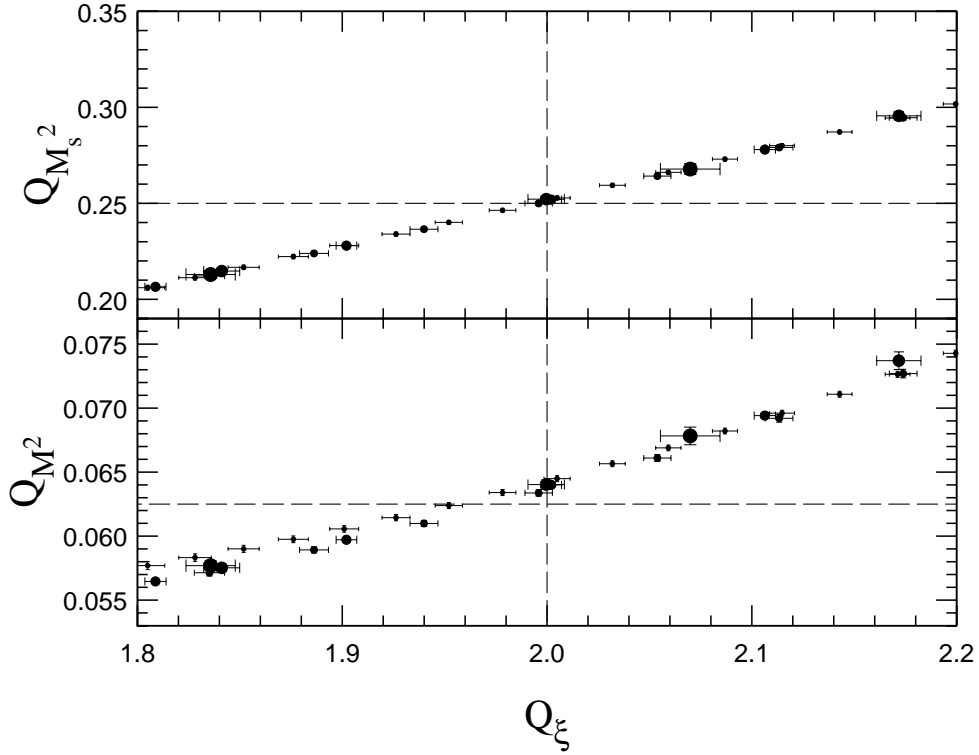


Figure 2: Quotients of  $M_s^2$  and  $M^2$  as a function of the quotient of  $\xi(L)$ . The horizontal straight lines correspond to mean-field behaviour. The symbol sizes are proportional to the lattice sizes.

The resulting exponents are shown in Table 2. After the name of the exponents their mean-field values are shown in square brackets [6]. The high accuracy reached on the measures of the  $\eta$  exponents, is due in part to the



$L_1, L_2$	$\nu[0.5]$	Staggered		Ferromagnetic	
		$\eta_\chi[0]$	$\eta_M[0]$	$\eta_\chi[2]$	$\eta_M[2]$
4,8	0.527(8)	0.009(4)	0.008(4)	1.959(8)	1.963(9)
6,12	0.524(6)	0.006(3)	0.008(3)	1.976(5)	1.981(6)
8,16	0.512(4)	0.008(2)	0.005(2)	1.968(3)	1.973(3)
10,20	0.491(7)	0.013(4)	0.001(4)	1.964(6)	1.969(6)
12,24	0.496(9)	0.007(5)	0.009(5)	1.960(8)	1.965(7)

Table 2: Estimations for the critical exponents of the AFM  $\mathbf{RP}^2$  model.

strong statistical correlation between  $Q_\xi$  and  $Q_{M^2}$ .

We obtain a value for  $\nu$  compatible with the mean-field prediction as well as for the magnetic exponents of the staggered sector. However, in the ferromagnetic channel, our exponents are close but not compatible with those given by mean-field theory.

Another possible interpretation of these values is that they could be *effective* critical exponents due to the presence of strong logarithmic corrections in the FM sector. To check this, let us suppose that there are logarithmic corrections only in the susceptibility. We do not take into account here the fact of possible logarithmic corrections to  $\xi$  as we use the values of the quotients measured at  $r = 2$  value. This is an approximation that holds for large  $L$ . We address to section §4 for a more complete treatment of the logarithmic corrections. So that

$$\chi \sim L^m (\ln L)^{\bar{m}}, \quad (17)$$

$$Q_\chi = \frac{\chi_{2L}}{\chi_L} = 2^m \left( \frac{\ln 2L}{\ln L} \right)^{\bar{m}} \equiv 2^m h_L^{\bar{m}}, \quad (18)$$

where  $m = \gamma/\nu$ . The effective exponents obtained with the standard FSS  $m'_L = \ln Q_\chi / \ln 2$ , can be written as

$$m'_L = m + \bar{m} \frac{\ln h_L}{\ln 2}. \quad (19)$$

The fit discarding the 4,8 pair for the ferromagnetic susceptibility gives

$$m = 0.09(9), \quad \bar{m} = -0.13(7), \quad \chi^2/\text{d.o.f.} = 0.04/2. \quad (20)$$

The fit (20) indicates that the exponents of the ferromagnetic channel in Table 2 are compatible with a mean-field situation with logarithmic corrections in the susceptibility.

### 3.3 Critical temperature

The thermodynamical critical temperature of our system can be estimated from the crossing points of the Binder cumulants for the different lattice sizes [12]. To obtain  $\beta_c(\infty)$ , we can extrapolate according to the formula

$$\beta_c(\infty) - \beta_c(L) \approx L^{-1/\nu}. \quad (21)$$

It should be noted that (21) is only a first approximation, because it does not include any logarithmic corrections. In section §4.3 we will be able to measure  $\beta_c(\infty)$  taking into account logarithmic corrections. However, it hardly modifies the value of the critical coupling obtained with this method.

We show in Fig. 3 the Binder cumulants for the staggered and ferromagnetic channels.

We have fitted the crossing points of the Binder cumulants of the  $L = 8$  lattice with lattices  $L = 12, 16, 20$  and  $24$ . The results of the fit are

<b>Staggered</b>	<b>Ferromagnetic</b>
$\beta_c(\infty) = -1.3426(3)$	$\beta_c(\infty) = -1.3421(6)$
$\chi^2/\text{d.o.f.} = 1.2/2$	$\chi^2/\text{d.o.f.} = 1.0/2$

Both values are compatible, and, as we expect one transition point, we take the value of  $\beta_c(\infty)$  with lower error, that is

$$\beta_c(\infty) = -1.3426(3). \quad (22)$$

Considering the crossing of the Binder cumulants of the  $L = 10$  lattice with the larger lattices scarcely change the numbers.

## 4 Logarithmic corrections of the $\text{RP}^2$ model

As we have previously shown, the values obtained for the critical exponents for the four dimensional AFM  $\text{RP}^2$  model, are compatible with those predicted by mean-field plus logarithmic corrections. They appear more clearly in the FM channel because in this sector there is no power-law divergence of

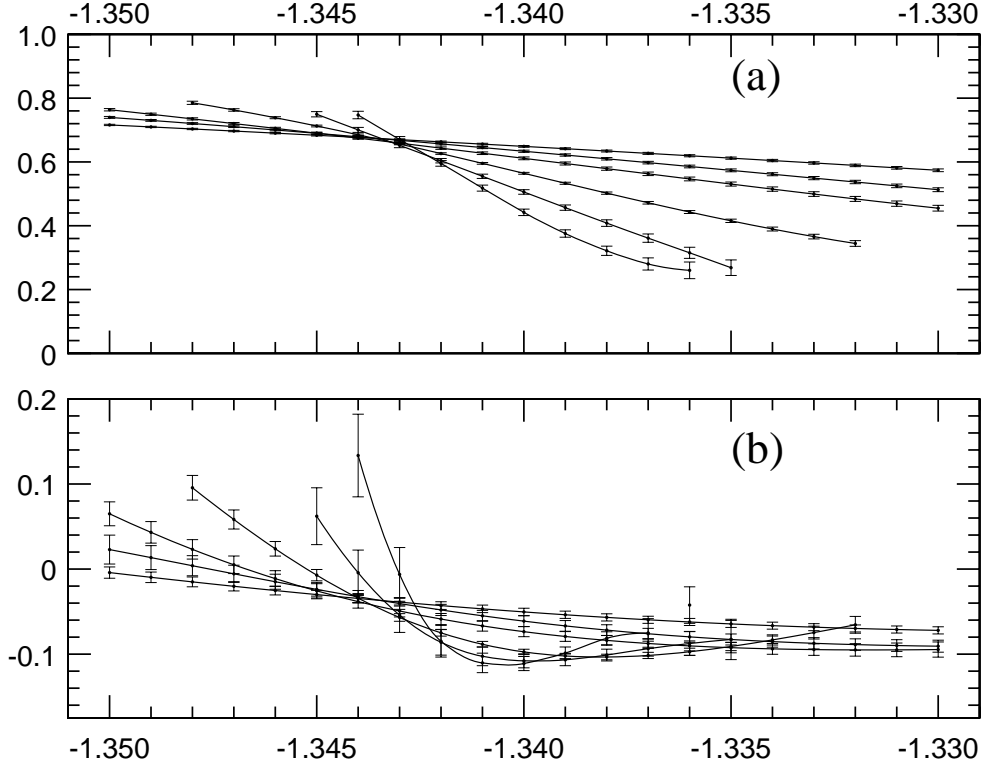


Figure 3: Binder cumulants for both sectors, (a) staggered, (b) ferromagnetic.

the susceptibility, while in the staggered sector the logarithmic divergence is added to a power-law one. We need a modification of the standard FSS to include these corrections.

#### 4.1 FSS with logarithmic corrections

Let us consider an observable  $O(t)$  whose behaviour near the critical point would have a logarithmic contribution

$$\langle O(t) \rangle \sim t^{-x} |\ln t|^{\bar{x}}. \quad (23)$$

We will follow [13] to take into account the logarithmic corrections: the scaling variable  $s(L, t)$  of (9) is now substituted by  $\xi(L, 0)/\xi(t)$ :

$$\langle O(L, t) \rangle = \langle O(t) \rangle \mathcal{F}_O \left( \frac{\xi(L, 0)}{\xi(t)} \right). \quad (24)$$

Formula (24) coincides with (9) below the upper critical dimension, where  $\xi(L, 0) \sim L$ , otherwise it can also take into account the logarithmic corrections to the finite volume correlation length.

Let us suppose that, to leading order,

$$\xi(L, 0) \sim L^{\hat{\alpha}} (\ln L)^{\hat{\beta}}, \quad (25)$$

where  $\hat{\alpha}$  and  $\hat{\beta}$  are two exponents that depend on the theory. For the  $O(N)$  models,  $\hat{\alpha} = 1$ ,  $\hat{\beta} = \frac{1}{4}$  [11]. The transition at finite  $L$  takes place when  $t$  is such that  $\xi(L, 0) \sim \xi(t)$ . If  $\xi(t) \sim t^{-\nu} |\ln t|^{\bar{\nu}}$ , then, employing (25),

$$t \sim L^{-\hat{\alpha}/\nu} (\ln L)^{(\bar{\nu}-\hat{\beta})/\nu}. \quad (26)$$

Below four dimensions  $\hat{\alpha} = 1$  and  $\hat{\beta} = \bar{\nu} = 0$ . Now, we have

$$s(L, t) \equiv \frac{\xi(L, 0)}{\xi(t)} \sim \frac{L^{\hat{\alpha}} (\ln L)^{\hat{\beta}}}{t^{-\nu} |\ln t|^{\bar{\nu}}}, \quad (27)$$

so that making use of (26), and with a change of variable similar to (11), we obtain

$$\langle O(L, t) \rangle = L^{\hat{\alpha}x/\nu} (\ln L)^{x/\nu(\hat{\beta}-\bar{\nu})+\bar{x}} f_O \left( \frac{\xi(L, t)}{L^{\hat{\alpha}} (\ln L)^{\hat{\beta}}} \right), \quad (28)$$

which is the equation analogue to (13). We follow now the same method as in the standard FSS case: we compute the quotient

$$Q_O = \frac{\langle O(2L, t) \rangle}{\langle O(L, t) \rangle} = 2^{\hat{\alpha}x/\nu} \left( 1 + \frac{\ln 2}{\ln L} \right)^{x/\nu(\hat{\beta}-\bar{\nu})+\bar{x}} \frac{f_O \left( \frac{\xi(2L, t)}{(2L)^{\hat{\alpha}} (\ln 2L)^{\hat{\beta}}} \right)}{f_O \left( \frac{\xi(L, t)}{L^{\hat{\alpha}} (\ln L)^{\hat{\beta}}} \right)}. \quad (29)$$

Measuring  $Q_O$  at the point  $t_L$  where

$$\frac{\xi(2L, t_L)}{\xi(L, t_L)} = 2^{\hat{\alpha}} h_L^{\hat{\beta}}, \quad (30)$$

with  $h_L \equiv 1 + \frac{\ln 2}{\ln L}$ , we find

$$Q_O(t_L) = 2^{\hat{\alpha}x/\nu} h_L^{x/\nu(\hat{\beta}-\bar{\nu})+\bar{x}}. \quad (31)$$

This is the new expression that substitutes (15) when there are logarithmic corrections, from it we can extract the exponent  $x/\nu$ . We no longer have to measure  $Q_O$  where the quotient of correlation lengths is 2, but instead where it equals  $2^{\hat{\alpha}} h_L^{\hat{\beta}}$ .

## 4.2 Staggered channel: renormalized four-point coupling

We proceed now to calculate the renormalized four-point coupling of our theory. The limit we are interested in is

$$g_R = \lim_{L \rightarrow \infty} g_R(L, \beta_c(\infty)). \quad (32)$$

The evolution of  $g_R$  at the critical temperature with  $L$  is shown in Table 3, where we have used the value for  $\beta_c(\infty)$  of (22).

$L$	Staggered	Ferromagnetic
8	3.16 (3)	-0.19 (3)
10	2.84 (4)	-0.19 (5)
12	2.61 (5)	-0.21 (5)
16	2.34 (4)	-0.23 (5)
20	2.08 (9)	-0.23 (11)
24	1.95 (13)	-0.15 (17)

Table 3: The renormalized four-point coupling  $g_R(L, \beta_c(\infty))$ .

When hyperscaling is violated by logarithms,  $g_R \sim |\ln t|^{-\bar{\rho}}$  [14]. Let us apply the FSS formula (24):

$$g_R(L, t) = g_R(t) \mathcal{F}_{g_R} \left( \frac{L^{\hat{\alpha}} (\ln L)^{\hat{\beta}}}{t^{-1/2} |\ln t|^{\bar{\nu}}} \right). \quad (33)$$

The scaling behaviour with  $L$  at the critical point is

$$g_{\text{R}}(L, 0) = |\ln t|^{-\bar{\rho}} \lim_{t \rightarrow 0} \mathcal{F}_{g_{\text{R}}} \left( \frac{L^{\hat{\alpha}} (\ln L)^{\hat{\beta}}}{t^{-1/2} |\ln t|^{\bar{\nu}}} \right). \quad (34)$$

We can eliminate the  $t$  dependence in (34) if  $\lim_{t \rightarrow 0} \mathcal{F}_{g_{\text{R}}}(z) \sim z^{-\bar{\rho}/\bar{\nu}}$ , which makes the  $|\ln t|^{-\bar{\rho}}$  factor disappear. After that, we employ relation (26) and get finally

$$g_{\text{R}}(L, 0) \sim (\ln L)^{-\bar{\rho}}, \quad (35)$$

which is also directly obtained from (28). We can obtain  $\bar{\rho}$  fitting the values of Table 3 to the functional form (35). The fit for all lattice sizes,  $L \geq 8$  yields (Fig. 4)

$$\bar{\rho} = 1.07(6), \quad \chi^2/\text{d.o.f.} = 0.8/4. \quad (36)$$

The result (36) implies triviality for the staggered sector of the AFM  $\mathbf{RP}^2$  model in four dimensions. The renormalized coupling goes to zero because of logarithmic corrections, exactly in the same way as in the ferromagnetic  $O(N)$  models, for which  $\bar{\rho} = 1$  [15].

The behaviour of the ferromagnetic channel is however rather different from that of the staggered sector. From the data of Table 3 we cannot conclude an asymptotic value for the renormalized coupling. We have seen above that the logarithmic corrections are very strong in this channel, and maybe for that reason, the renormalized coupling is very hard to measure. To conclude about the triviality of this sector, we will try to study in full detail the logarithmic corrections of this channel, following the FSS analysis derived in section §4.1.

## 4.3 Computation of the logarithmic corrections

### 4.3.1 Correlation lengths

To parameterize the logarithmic corrections, we need a lot of, in principle, unknown exponents:  $\hat{\alpha}, \hat{\beta}, \bar{\nu}, \dots$ . Therefore it will be necessary to make a few assumptions about these exponents.

The most important exponents are  $\hat{\alpha}$  and  $\hat{\beta}$ , (25), because we have to measure at the points which satisfy (30). Assuming a mean-field plus logarithmic corrections scenario, we expect  $\hat{\alpha} = 1$ . To find out the value of

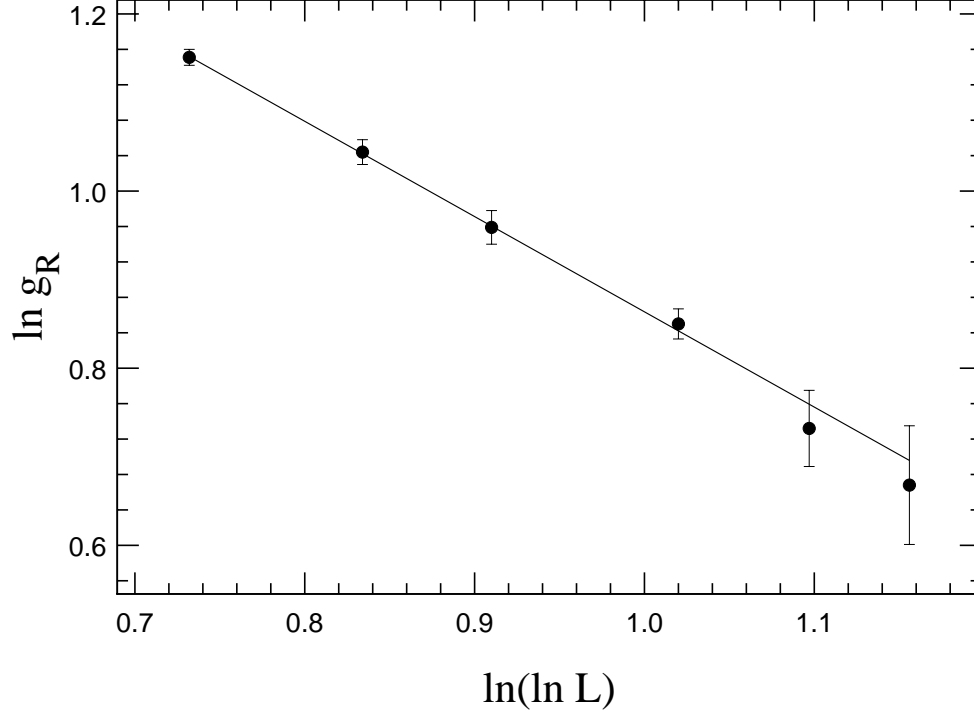


Figure 4: Fit of  $g_R(L)$  (staggered channel) at  $\beta_c(\infty)$ .

exponent  $\hat{\beta}$ , we use the prediction (25) for the correlation length in a finite lattice at  $\beta_c(\infty)$ :

$$\ln \frac{\xi_L}{L} = C + \hat{\beta} \ln(\ln L). \quad (37)$$

We have performed the fit for both correlation lengths, the ferromagnetic ( $\xi^{\text{FM}}$ ), and the staggered ( $\xi$ ) one. In order to monitorize subleading effects, we have compared the fits with  $L \geq 8$ , and  $L \geq 10$ . We have found that the fit for  $\xi^{\text{FM}}$  is more stable with growing lattice sizes. In Table 4, we show the fit parameters. The infinite volume critical coupling, and the fit-parameter errors have been estimated from the increment in one unit of the  $\chi^2$  function. Comparing with the previous determination of the critical coupling (22), both determinations are consistent and of similar accuracy, although logarithmic corrections to scaling were not considered previously. Our value for the exponent  $\hat{\beta}$  is consistent with the predicted value for the  $O(N)$  models,

	Fit	$\chi^2/\text{d.o.f.}$	$\hat{\beta}$	$\beta_c(\infty)$
$\xi$	$L_{\min} = 8$	2.1/3	0.21(2)	-1.3423(3)
	$L_{\min} = 10$	0.3/2	0.16(3)	-1.3424(6)
$\xi^{\text{FM}}$	$L_{\min} = 8$	0.8/3	0.22(4)	-1.3425(3)
	$L_{\min} = 10$	0.2/2	0.17(8)	-1.3424(3)

Table 4: Fits for the logarithmic corrections to the correlation lengths at the critical point.

$\hat{\beta} = 0.25$ , specially the obtained from the ferromagnetic correlation length.

#### 4.3.2 Magnetic operators

We shall try to control other logarithmic corrections by making use of the results of section §4.1. From (29) we know that

$$Q_\chi = 2^{\hat{\alpha}\gamma/\nu} h_L^{\gamma/\nu(\hat{\beta}-\bar{\nu})+\bar{\gamma}}, \quad (38)$$

when measuring the quotient at the point which verifies the condition (30). The logarithmic corrections are given in terms of the unknown exponents  $\bar{\nu}$  and  $\bar{\gamma}$ , but, as we assume mean-field exponents,  $\gamma = 0$ ,  $\nu = 0.5$ , we can reduce (38) to

$$\ln Q_\chi = \bar{\gamma} \ln h_L. \quad (39)$$

In a similar way, if we take the magnetization,

$$Q_M = 2^{-\hat{\alpha}\beta/\nu} h_L^{-\beta/\nu(\hat{\beta}-\bar{\nu})+\bar{\beta}}, \quad (40)$$

or (mean-field:  $\beta = 1$ )

$$\ln Q_M = -2 \ln 2 + \bar{\kappa} \ln h_L, \quad (41)$$

where  $\bar{\kappa} = -2(\hat{\beta} - \bar{\nu}) + \bar{\beta}$ . Therefore, from every pair of lattices  $L, 2L$ , we can obtain the exponents  $\bar{\gamma}, \bar{\kappa}$ . This is shown in Table 5.

Notice that the  $\bar{\gamma}$  and  $\bar{\kappa}$  values are very close to their corresponding values for the magnetization in O(3) and O(4) models. It should be understood that these values ( $\bar{\gamma} \sim 0.5$  and  $\bar{\kappa} \sim 0.25$ ) are calculated for the order parameter on the fundamental (vectorial) representation of the O( $N$ ) group. However



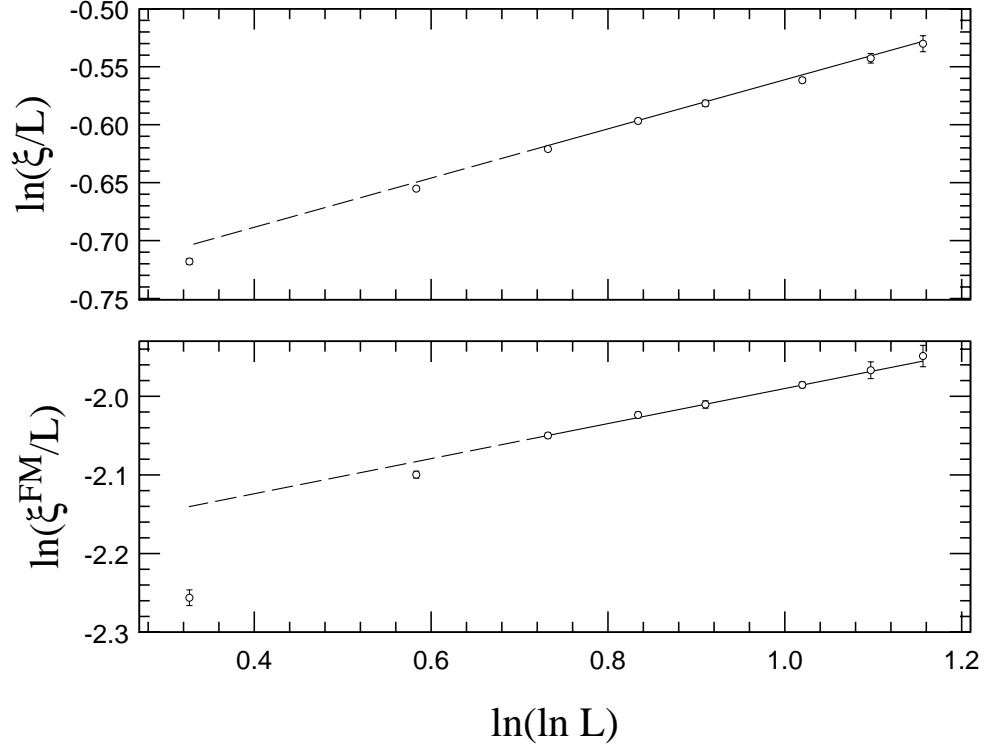


Figure 5: Determination of the exponent  $\hat{\beta}$  of the FSS formulas from the behaviour of  $\xi^{\text{FM}}$  and  $\xi$  measured at the mean values of  $\beta_c$  from the fits with  $L \geq 8$ .

the critical exponents of the FM magnetization in the  $\mathbf{RP}^2$  model and those of the order parameter in the tensorial representation for the  $O(N)$  models are the same at the mean-field level. Let us remark that the  $\bar{\gamma} \sim 2\bar{\kappa}$  just means that  $\langle M^2 \rangle \sim \langle M \rangle^2$ .

This final result from the FM sector completes the conclusion that we obtained after examining the renormalized coupling of the staggered sector in §4.2: the  $\mathbf{RP}^2$  model is trivial due to the logarithmic corrections to the mean-field behaviour.

The question of the SSB pattern remains unsolved as the ferromagnetic susceptibility is only logarithmically divergent. We recall that the power-law behaviour was crucial to check the symmetry breaking in three dimensions [6].

$L_1, L_2$	$\bar{\gamma}$	$\bar{\kappa}$
4,8	0.45 (2)	0.23 (6)
6,12	0.45 (1)	0.22 (6)
8,16	0.49 (1)	0.24 (3)
10,20	0.53 (2)	0.26 (1)
12,24	0.52 (3)	0.26 (2)

Table 5: Exponents of the logarithmic corrections of the FM sector of the  $\mathbf{RP}^2$  model.

## 5 Conclusions

We have examined the triviality question of the four dimensional AFM  $\mathbf{RP}^2$  model, which presents a second order transition. A very interesting feature of this model is that it presents two different order parameters. A detailed study of these two sectors reveals that the model has a trivial continuum limit. We have been able to calculate explicitly the logarithmic corrections to the mean-field behaviour by means of a FSS analysis also valid at the critical dimension of the model.

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## A The overrelaxed algorithm

Overrelaxation is a local microcanonical update algorithm. It makes the maximum change in the variable at a given point without modifying the energy. We will now describe how this method works in our model. It is easy to see that

$$(\mathbf{v}_i \cdot \mathbf{v}_j)^2 = \text{tr} \mathbf{T}_i \mathbf{T}_j + \frac{1}{3}, \quad (42)$$

so that the change  $\mathbf{T}_i \rightarrow \mathbf{T}'_i$  must be such that the  $\text{tr}\mathbf{T}_i\mathbf{N}$  is conserved, where  $\mathbf{N}$  is the sum of the tensors at the neighbour points of  $i$ . If we set  $\mathbf{T}' = R\mathbf{T}R^{-1}$ , the conservation of the energy is then expressed by

$$[R, \mathbf{N}] = 0. \quad (43)$$

Besides, we must ensure that the new tensor belongs to  $\mathbf{RP}^2$ , so that the change in  $\mathbf{T}$  is associated with a change in  $\mathbf{v}$ :  $\mathbf{v}' = R\mathbf{v}$ . As the vectors are normalized, this puts also the condition of unitarity on the matrix  $R$ .

In order to fulfill these two conditions, let us write  $\mathbf{N}$  as  $\mathbf{N} = U\mathbf{\Lambda}U^{-1}$ , where  $\mathbf{\Lambda}$  is the matrix of eigenvalues of  $\mathbf{N}$ , and define  $C = U^{-1}RU$ . Then, the updating conditions, written in terms of the matrix  $C$  are

$$[C, \mathbf{\Lambda}] = 0 \quad , \quad C^{-1} = C^+. \quad (44)$$

As  $\mathbf{\Lambda}$  is a diagonal matrix (44) implies that  $C$  has to be also diagonal, and  $C^2 = \mathbf{1}$ , which means that its three eigenvalues will be  $\pm 1$ . We have reduced our updating process to a choice of the matrix  $C$ . Here enters the second characteristic of the overrelaxed algorithm: the change in the vector  $\mathbf{v}$  should be maximum, which can be achieved by minimizing the value of the squared scalar product

$$\mathcal{A} = (\mathbf{v} \cdot \mathbf{v}')^2 = (\mathbf{v} \cdot R\mathbf{v})^2 = (\tilde{\mathbf{v}} \cdot C\tilde{\mathbf{v}})^2, \quad (45)$$

where

$$\tilde{\mathbf{v}} = U^{-1}\mathbf{v} \equiv (x_1, x_2, x_3). \quad (46)$$

To do the update, we then have to take the three numbers  $c_i = \pm 1$  that minimize the quantity

$$\mathcal{A} = (c_1x_1^2 + c_2x_2^2 + c_3x_3^2)^2. \quad (47)$$

To sum up, the overrelaxed algorithm consists of calculating the matrix  $\mathbf{N}$  of nearest neighbours, its eigenvectors to obtain  $U$ , and then looking for the minimum of the combinations in (47).

It is easy to see that this algorithm verifies detailed balance: if we make  $\mathbf{v} \rightarrow \mathbf{v}' \rightarrow \mathbf{v}''$ , so that  $\mathbf{v}'' = UC'\tilde{\mathbf{v}}'$  and  $\tilde{\mathbf{v}}' = C\tilde{\mathbf{v}}$ , we have to minimize

$$(\tilde{\mathbf{v}}' \cdot C'\tilde{\mathbf{v}}')^2 = (\tilde{\mathbf{v}} \cdot CC'C\tilde{\mathbf{v}})^2, \quad (48)$$

and the last expression was minimized by  $C$ , so that  $C = CC'C$  or  $C' = C$ . Therefore  $\mathbf{v}'' = \mathbf{v}$ .

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